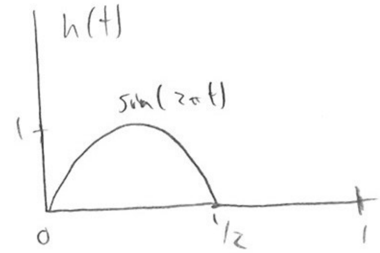
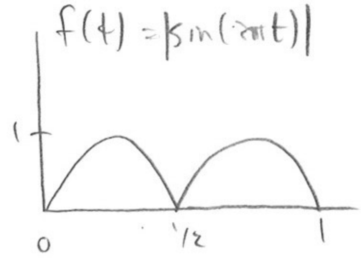
## rectified sine waves

Find the FS of a half rectified and full rectified sine wave.

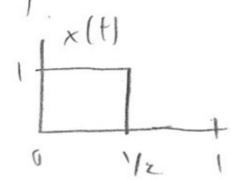



First of all, note that $f(t) = h(t) + h(t - \frac{1}{2})$. If we find $H[k]$, it is trivial to find $F[k]$. We will find $H[k]$ using 3 different methods.

**method 1:** from known FS

Say we know the FS of a 50% duty cycle square wave.




$$X[k] = \frac{1}{2}\,\mathrm{sinc}\left(\frac{k}{2}\right) e^{-\pi j k/2} = \begin{cases} \frac{1}{2}\sin\left(\frac{\pi k}{2}\right)\frac{2}{\pi k}(-j)^k & k \text{ odd} \\ 0 & k \text{ even} \\ 1/2 & k = 0 \end{cases}$$

$$= \begin{cases} \frac{1}{j\pi k} & k \text{ odd} \\ 0 & k \text{ even} \\ 1/2 & k = 0 \end{cases}$$

$$\underset{\downarrow}{e^{2\pi j(1)t/1}} \qquad \underset{\downarrow}{e^{2\pi j(-1)t/1}}$$

$$h(t) = x(t)\sin(2\pi t) = x(t)\left(e^{2\pi j t} - e^{-2\pi j t}\right)\frac{1}{2j}$$

$$\Rightarrow H[k] = \frac{1}{2j}\left(X[k-1] - X[k+1]\right) \quad \text{(modulation rule)}$$

$$= \frac{1}{2j}\left(\begin{cases} \frac{1}{j\pi(k-1)} & k-1 \text{ odd} \\ 0 & k-1 \text{ even} \\ 1/2 & k-1 = 0 \end{cases} - \begin{cases} \frac{1}{j\pi(k+1)} & k+1 \text{ odd} \\ 0 & k+1 \text{ even} \\ 1/2 & k+1 = 0 \end{cases}\right)$$

$$= \frac{1}{2j}\begin{cases} 0 & k \text{ odd} \\ \pm\frac{1}{2} & k = \pm 1 \\ \frac{1}{j\pi(k-1)} - \frac{1}{j\pi(k+1)} & k \text{ even} \end{cases}$$

$$\frac{1}{k-1} - \frac{1}{k+1} = \frac{(k+1)-(k-1)}{(k-1)(k+1)} = \frac{2}{k^2-1}$$

$$\boxed{= \begin{cases} \pm\frac{1}{4j} & k = \pm 1 \\ 0 & k \text{ odd} \\ \frac{1}{\pi}\,\frac{-1}{k^2-1} & k \text{ even} \end{cases} = H[k]}$$

rectified sine wave cont.

method 2: direct integration

$$H[k] = \frac{1}{1}\int_0^{1/2} \sin(2\pi t)\, e^{-2\pi j k t/1}\, dt$$

$$= \frac{1}{2j}\int_0^{1/2} \left(e^{+2\pi j t} - e^{-2\pi j t}\right) e^{-2\pi j k t}\, dt$$

$$\int_0^{\frac{1}{2}} e^{\pm 2\pi j t}\, e^{-2\pi j k t}\, dt = \int_0^{1/2} e^{2\pi j(-k \pm 1)t}\, dt$$

$$= \left.\frac{e^{2\pi j(-k\pm 1)t}}{2\pi j(-k\pm 1)}\right]_{t=0}^{1/2} = \frac{e^{2\pi j(-k\pm 1)/2} - 1}{2\pi j(-k\pm 1)}$$

$$H[k] = \frac{1}{2j}\left(\frac{e^{2\pi j(-k+1)/2} - 1}{2\pi j(-k+1)} - \frac{e^{2\pi j(-k-1)/2} - 1}{2\pi j(-k-1)}\right)$$

$$e^{-2\pi j(k\mp 1)/2} = e^{-\pi j(k\mp 1)} = (-1)^{k\mp 1} = -(-1)^{k}$$

$$= \frac{-1}{4\pi}\left(\frac{-(-1)^k - 1}{-k+1} - \frac{-(-1)^k - 1}{-k-1}\right)$$

$$= \frac{1}{4\pi}\left((-1)^k + 1\right)\left(\frac{1}{k+1} - \frac{1}{k-1}\right)$$

$$= \frac{2}{4\pi}\left(\frac{(k-1)-(k+1)}{(k+1)(k-1)}\right) \quad k \text{ even}$$

$$= \frac{1}{2\pi}\,\frac{-2}{k^2-1} = \begin{cases} \dfrac{1}{\pi(1-k^2)} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

Beware! $k = \pm 1$ is a special case! ($\frac{0}{0}$ = undef, and you <u>cannot</u> use L'Hopital's rule)

$$H[\pm 1] = \int_0^{1/2} \sin(2\pi t)\, e^{-2\pi j(\pm 1)t}\, dt$$

$$= \int_0^{1/2} \frac{1}{2j}\left(e^{2\pi j t} - e^{-2\pi j t}\right) e^{-2\pi j(\pm 1)t}\, dt$$

$$= \int_0^{1/2} \frac{1}{2j}\begin{Bmatrix} 1 - e^{-4\pi j t} & k = +1 \\ e^{+4\pi j t} - 1 & k = -1 \end{Bmatrix} dt =$$

rectified sinusoid cont

method 2 cont

$$H[1] = \frac{1}{2j}\int_0^{1/2}\left(1 - e^{-4\pi j t}\right)dt \quad \rightarrow \quad H[\pm 1] = \frac{1}{2j}\int_0^{1/2} \pm\left(1 - e^{\mp 4\pi j t}\right)dt$$

$$H[-1] = \frac{1}{2j}\int_0^{1/2} -\left(1 - e^{+4\pi j t}\right)dt$$

$$\int_0^{1/2} e^{\pm 4\pi j t}\,dt = \int_0^{1} e^{\pm 2\pi j t}\,dt = 0 \quad \left(\text{think } \int_0^{2\pi}\cos(t)\,dt = \int_0^{2\pi}\sin(t)\,dt = 0\right)$$

$$H[\pm 1] = \frac{1}{2j}\int_0^{1/2}(\pm 1)\,dt = \pm\frac{1}{4j}$$

So finally:
$$\boxed{H[k] = \begin{cases} 0 & k \text{ odd} \\ \frac{1}{\pi(1-k^2)} & k \text{ even} \\ \pm\frac{1}{4j} & k = \pm 1 \end{cases}}$$

method 3 Poisson Summation

$$x(t) = \Pi\left(\frac{t - 1/4}{1/2}\right)\sin(2\pi t) = \begin{cases} \sin(2\pi t) & 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

$$\Pi(t) \xrightarrow{\mathcal{F}} \operatorname{sinc}(f)$$

$$\Pi\left(\frac{t}{1/2}\right) \xrightarrow{\mathcal{F}} \frac{1}{2}\operatorname{sinc}\left(\frac{f}{2}\right)$$

$$\Pi\left(\frac{t - 1/4}{1/2}\right) \xrightarrow{\mathcal{F}} \frac{1}{2}\operatorname{sinc}\left(\frac{f}{2}\right)e^{-2\pi j f\left(\frac{1}{4}\right)} = \frac{1}{2}\operatorname{sinc}\left(\frac{f}{2}\right)e^{-\pi j f/2}$$

$$\Pi\left(\frac{t - 1/4}{1/2}\right)\sin(2\pi t) = \Pi\left(\frac{t - 1/4}{1/2}\right)\frac{1}{2j}\left(e^{+2\pi j t} - e^{-2\pi j t}\right) \xrightarrow{\mathcal{F}} \frac{1}{4j}\operatorname{sinc}\left(\frac{f-1}{2}\right)e^{-\pi j(f-1)/2} - \frac{1}{4j}\operatorname{sinc}\left(\frac{f+1}{2}\right)e^{-\pi j(f+1)/2}$$

$$h(t) = \sum_{n=-\infty}^{\infty} x(t - n\cdot 1) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{1}X\left(\frac{k}{1}\right)}_{H[k]} e^{2\pi j k t/1}$$

$$H[k] = X(k) = \frac{1}{4j}\left(\operatorname{sinc}\left(\frac{k-1}{2}\right)\underbrace{e^{\pi j/2}}_{=j} - \operatorname{sinc}\left(\frac{k+1}{2}\right)\overbrace{e^{-\pi j/2}}^{=-j}\right)\cdot\underbrace{e^{-\pi j k/2}}_{=(-j)^k}$$

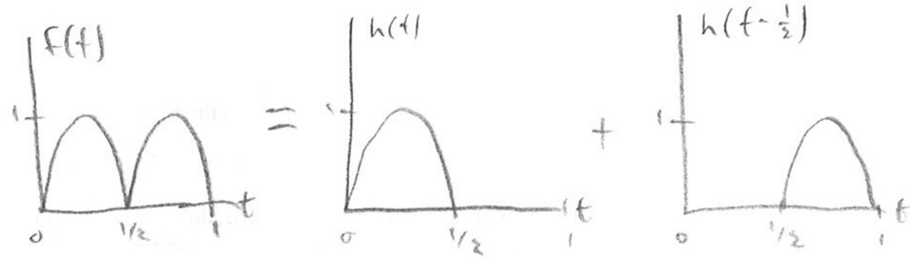
$$\boxed{H[k] = \frac{(-j)^k}{4}\left(\operatorname{sinc}\left(\frac{k-1}{2}\right) + \operatorname{sinc}\left(\frac{k+1}{2}\right)\right)}$$

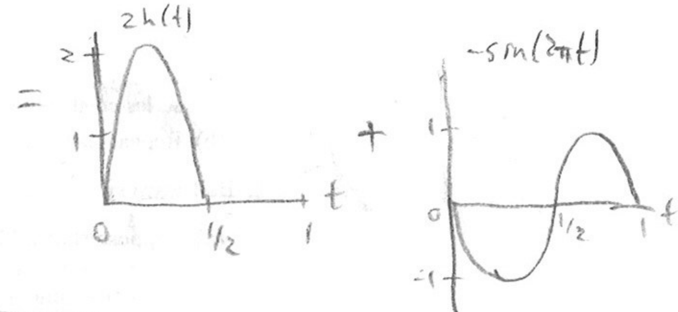
we could simplify more, but I've already done that in methods 1 & 2.

rectified sine waves cont.

We have $H[k]$, the FS coefficients for the half-wave-rectified sine wave. We can find the FS coeffs. of the full-wave rectified wave relatively easily:

$$f(t) = h(t) + h\left(t - \tfrac{1}{2}\right)$$
$$= 2h(t) - \sin(2\pi t)$$




$$F[k] = H[k] + e^{-2\pi j k\left(\frac{1}{2}\right)} H[k]$$
$$= H[k]\left(1 + (-1)^k\right)$$
$$= \begin{cases} 2H[k] & k \text{ even} \\ 0 & k \text{ odd} \end{cases} = \boxed{\begin{cases} \frac{2}{\pi(1-k^2)} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}}$$

$$F[k] = 2H[k] - \frac{1}{2j}\left(\delta[k-1] - \delta[k+1]\right) \qquad \left(e^{+2\pi j k_0 t / T_0} \xrightarrow{\mathcal{F}_{FS}} \delta[k-k_0]\right)$$

$$= \begin{cases} \frac{2}{\pi(1-k^2)} & k \text{ even} \\ 0 & k \text{ odd} \\ \pm\frac{1}{2j} & k = \pm 1 \end{cases} - \begin{cases} \pm\frac{1}{2j} & k = \pm 1 \\ 0 & \text{else} \end{cases} = \boxed{\begin{cases} \frac{2}{\pi(1-k^2)} & k \text{ even} \\ 0 & \text{else} \end{cases}}$$

It was important to observe that it is easy to get $f(t)$ in terms of $h(t)$, but not the other way around. That's why I did $h(t)$ first.